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ADVANCES IN MATHEMATICAL MODELING: BOUNDARY VALUE PROBLEM SOLUTIONS FOR NONLINEAR FRACTIONAL FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract
The field of fractional calculus has seen remarkable developments in recent years, accompanied by the application of fractional differential equations in diverse domains such as automatic control theory, biology, and viscoelasticity. These equations offer a more accurate representation of real-world phenomena by considering not only the current state of a system but also its past state and the rate of state change. Functional differential equations, in particular, have found extensive applications in signal recognition, economics, physics, and other fields

Keywords: Fractional calculus, Functional differential equations, fixed point theorem, Caputo fractional derivative, Boundary value problems.

1. Introduction

In recent years, with the further development of the theory of fractional calculus and the application of fractional differential equations in the fields of automatic control theory, biology and viscoelasticity[1-2], which has received extensive attention. Most of the mathematical models established to solve practical problems are completed in an ideal state. It is assumed that the changing laws of things are only related to the current state. However, in actual problems, the future behavior of the system depends not only on the current state. At the same time, it may also be affected by the past state or the rate of state change. Therefore, it is considered that the functional differential equation can describe the objective world more accurately, and it has a wide range of applications in signal recognition, economics, physics and other fields[3].

In this paper, we use the fixed point theorem of cone extension and cone compression to study the following nonlinear fractional functional integro-differential equation boundary value problems with two fractional derivative terms

□$^cD_0$□□[^cD u t_0□□ ( ) □h t u(, $_t$)]□ f t u Qu t(, $_t$, ( )) □ 0, t □(0,1),
□
□□□□u t^cD u( )$_0$□□ □(0)□( )t□ □, a, u□□(0) □ □g u(( ))□ , t□ □[□,0], (1)
□u(1) □ □^1k s u s(, ( ))d ,s
□□ 0

where^cD0□□ is the Caputo fractional derivative operator, 0□ □□ 1, 2 □ □□ 3, 0 □ □□ 1, □□0, u_t
□u_t( )□ □u t(□□) , □ □□ □[,0] , g□C(ℝ□, ℝ□) , C_1 □ □C([□,0], ℝ) , □□ □C([□,0], ℝ□) ,
t
□(0) 0□ , q_0 □ sup0 1□ □t □0q t s(,)ds ,

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q□C([0,1]□[0,1],ℝ□), C₁□ □ □□□ □□ □ □C₁ :( ) □ 0, □ □[,0]□, a□ℝ□,

t a □ h(0,□) , f □C([0,1]□ □C₁□ ℝ□,ℝ□), Qu t( ) □□0 q t s u s(,) ( )ds, h□C([0,1]□C₁□, ℝ□) , k□C([0,1]□ □, □) , □ℝ□ ℝ□ □□ℝ[0,) .

For any □□C₁, we define norm||□||C1 □ sup |□□( ) |, then C₁ is a Banach space. We denote □ □□□[,0]

that E□ □ □□u C[□,1]:u(0) □ 0□ and endowed with the norm || u ||E □ sup | u t( ) |, then (E,|| ||)□
t□□[□,1]
is a Banach space. And we also denote that E₀ □ □ □{z C[□,1]: z t( ) □ □ □0,t [□,0]} and endowed with the norm || z ||□

sup | z t( ) |□ sup | z t( ) | , then E₀ is a Banach space, E₀ is a subset of E .
t□□[□,1]t□[0,1]

2. Preliminaries

Lemma 2.1 If the function u is the solution of the boundary value problem (1.1), then the function u satisfies the following integral equation

□□ ¹G t s f s u Qu s(,) (, s,( ))ds □□0¹H t s h s u(,) (, s)ds □ □(a h(0,□))□
□0
u t( ) ——— □□□□ t(□t□1) – □t□0¹k s u s ²) ¹2 g u(( ))□ , t□(0,1), (2) (, ( ))ds□ □(t t
□ □ □□
□□( )t , t□ □[□,0],
□□
where

1□□t(1□s)□□□ □1 □ □(ts)□□□ □1, 0 □ □ □s t 1,

G t s(,) □□ □(□ □) □□t(1□s)□□□ □1, 0 □ □ □t s 1, (3)
1 □□t(1□s)□□1 □ □(ts)□□1, 0 □ □ □s t 1,
H t s(,) □——□(□) □□□t(1□s)□□1, 0 □ □ □t s 1. (4)

Proof: Let u t( ) be the solution to the boundary value problem (1.1), then for any t□[0,1] , the general solution of fractional differential equation cD₀□□[cD u t₀□□ ( )□h t u(, t)]□ f t u Qu t(, t, ( )) □ 0 is given by cD u t₀□□ ( ) □□ ——(1□) □0t (t □s)□□1f s u Qu s(, s, ( ))ds□h t u(, t) □c₀ . The boundary
□
condition cD u₀□□ (0) □□a , we can obtain that c₀ □ h(0,□)□a , so
u t( ) □□———1□0t (t □s)□□□ □1f s u Qu s(, s, ( ))ds □——□(□) □0t (t □s)□□1h s u(, s)ds
1
□ □(□ □)
From the
t□ 2.

□ ((0*h* ,□) □*a*)______□ □ □c_1 *c t*$_2$ *c t*$_3$
□ □(□ 1)
boundary condition *u*(0) □□(0) □ 0 implies that c_1 □ 0 . Thus,
______ 1 *t*□□□ □1*f s u* ____ *Qu s*(, *s*, ( ))ds □ 1 □0*t* (*t* □*s*)□□1*h s*
u(, *s*)d*s*
u t( ) □□ □0 (*t* □*s*) □(□)
□ □(□ □)
c t ______
□ ((0*h* ,□) □*a*) *t*□ □ □*c t*$_2$ $_3$ 2.
□ □(□ 1)
1 *t* ______ □□□ □3 *f s u Qu s*(, *s*, ______ ( ))ds □ 1 □ *u*
t□□( ) □ □ (□0 (*t* □*s*) □(□□ 2)
□ □ □□□ 2)

□0*t* (*t* □*s*)□□3*h s u*(, *s*)ds □ ((0*h* ,□) □*a*) ______□(*t*□ □□□2 1) □ 2c_3.
g u(( ))□
The boundary condition *u*□□(0) □ □*g u*(( ))□ , we can obtain thatc_3 □□ ______ . Thus, 2
u t( ) □ ______(□1) □0*t* (*t* □*s*)□□□ □1 *f s u Qu s*(, *s*, ( ))ds□ ____ □(1□) □0*t* (*t*
□*s*)□□1□
□ □□ □
t□ 1 2 (( )).□ *h s u*(, *s*)ds□ ((0*h* ,□) ______ – □*a*) □ □*c t*$_2$ *t g u*
□ □(□ 1) 2

1
From the boundary condition *u*(1) □□0*k s u s s*(, ( ))d , thus,
□ ______ □(□ □1) 1 □□□ □1*f s u* ____ *Qu s*(, *s*, ( ))ds□ 1 □01(1□*s*)□□1*h s u*(, *s*)ds c_2 □
□0 (1□*s*) □()□
Thus we can

1 1 1
□((0,)*h* □□*a*)(1) □ 2 *g u*(( ))□ □□0 *k s u s*(, ( ))d .*s*
□ □□
get that
(1) *t* ______ □□□ □1*f s u Qu s s*(, *s*, ( ))d ____ □ 1 □0*t* (*t s*□)□□1*h s u*(, *s*)ds *u t*( ) □□ □0 (*t s*□
) □()□
□ □□ □ *t* 1 1*f s u Qu s s*(, *s*, ( ))d □ *t* □1(1□*s*) 1*h s u*(,)ds
□□ □(□ □) □0 (1□*s*)□□□ □ □( )□ 0 □□ *s*
□ ______ ______ *t* *t*□ □ (*a* *h*□ ____ (0,□))□ □(*t t*2) _1 *g u*(( ))□ □*t k s u s s*□1 (, (
))d When
□ □(□ 1) 2 0
1 1 1
□□0 *G t s f s u Qu s*(,) (, *s*,( ))ds□□0 *H t s h s u*(,) (, *s*)ds *t k s u s s*□ □0 (, ( ))d
□ ______*t t*□ □ (*a h*□ (0,□))□ □(*t t*2) _1 *g u*(( )).□
□ □(□ 1) 2

t□ □[□,0] , u t( ) □□( )t . So
□□□$_0^{1}G$ t s f s u Qu s(,) (, $_s$,( ))ds□□0^1 H t s h s u(,) (, $_s$)ds□t□0^1k s u s(, ( ))ds
u t( ) □ □□□□ t □t□ (a□h(0,□)) □ □(t t^2) _^{1} g u(( ))□ , t□(0,1), On the
□ □ □(□ 1) 2
□□( )t , t□ □[□,0].
□□
contrary, if (2.1) holds, it is easy to prove that u t( ) is the solution of boundary value problem (1.1).

Lemma2.2 For all □□, □(0,1) and □ □ □□□^{1} □ □ , then

(1) G t s(,) is continuous and 0 □G t s(,) □ p s_1( ) for t s, □[0,1]; G t s(,) □ (□□□ □□^{1})p s_1( ) for t□[□□,], s□[0,1];

(2) H t s(,) is continuous and 0 □ H t s(,) □ p s_2( ) for t s, □[0,1]; H t s(,) □ □(□□□□^{1})p s_2( ) for t□[□□,], s□[0,1];

(1□s)□□□ □1 (1□s)□□1
Where p s_1( ) □______, p s_2( ) □______.
□ □(□ □) □( )□

3. Main results

Supplement the definition of function □( )t , let t□[0,1] , □( ) 0t □ , then □□E . Making a transformation u t( )□□( )t □ z t( ), then for any t□[0,1], it is easy to show that u_t □ □□$_t$ z_t and the integral equation (2) equivalent to the integral equation

□□□$_0^{1}G$ t s f s(,) (,□$_s$ □ z Qz s_s,( ))ds□□0^1 H t s h s(,) (,□$_s$ □ z_s)ds
□□ 1 (t □t□)(a□h(0,□))(t □t^2)
z t( ) □ □□ t□0 k s z s(, ( ))ds□□ □(□ 1) □ $_2$$g$ z(( ))□ ,t□(0,1), (5)
□
□0, t□ □[□,0].
□□

Let P□□ □□zE_0 : z t( ) □ □□0,t [□,1], min$t$□[□□,] z t( ) □ □(□ □□□^{1}) || z ||□ ,
where □□(0,1),□ □ □□□^{1} □ □ . Obviously, P□□ E_0 is a cone, which is for any x y, □E x_0, y if and only y□ □x P□. Then (E_0,) is a semi-ordered Banach space. We define operator T P: □□ E_0 as

□□□$_0^{1}G$ t s f s(,) (,□$_s$ □ z Qz s_s,( ))ds□□0^1 H t s h s(,) (,□$_s$ □ z_s)ds
Tz t( ) □ □□□□□ t□01k s z s(, ( ))ds□ (t □t□□ □)((□a□h1)(0,□)) (t □2t2) (6) □ g z(( ))□ ,t□(0,1),
□0, t□ □[□,0].
□□

Lemma 3.1 Assume that f h, satisfies the f □C([0,1]□ □C_1□ ℝ□,ℝ□) , h□C([0,1]□C_1□, ℝ□) conditions, then the operator T P: □□P□ is completely continuous.

Lemma 3.2 (See[1]) Let E be a Banach space, and let P □ E be a cone in E .

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Assume □ □ □$_1$, $_2$ P are two bounded open subsets of E with 0□□ □□ □□$_1$ 1 $_2$, and let T P:(□ □ □$_2$ \ $_1$) P be a completely continuous operator such that either

(1) $||Tz||$ $||$□ $z||$,□ □z P □□$_1$; $|Tz|$| $||$□ $z||$,□ □z P □□$_2$;
(2) $||Tz||$ $||$□ $z||$,□ □z P □□$_1$; $|Tz|$| $||$□ $z||$,□ □z P □□$_2$.

Then T has a fixed point in P(□ □$_2$ \ $_1$) .

For convenience, we denote

f0 □ limsup sup $f\,t(\,,$□$,\,)v$, h0 □ limsup sup $h\,t(\,,$□$)$,
$||$□$||C$1□ □v0□ t□[0,1] $||$□$||C$1 □v $||$□$||C$1□0□ t□[0,1] $||$□$||C$1
0limsup sup $k\,t\,v(\,,\,)$, g0 □ limsup $g\,v(\,)$, k□ □ liminf inf $k\,t\,v(\,,\,)$.
k □ v□0□ t□[0,1] v v□0□ v v□□□ □t [0,1] v

Theorem 3.1 Suppose there are constants $N\,N_1$, $_2$, $N\,N_3$, $_4$, N_5 □ 0 , so that f^0 □ $N\,h_1$, ^{0} □ N_2, k^0 □ N_3, g^0 □ N_4, $k_□$ □ N_5 are established. If □□( ) □ 0, N1(1□q0) □ N2 □ □N3 N4 □1, N5 □1, □ □ □(□ □ 1) □ □(□ 1)
then there is a constant b□0, when $a^{□h(0,□)}$ □b , the boundary value problem (1.1) has at
□ □(□ 1)
least one positive solution.

Proof: Selection cone $P_□$, 0 □□ □ □□□1, □□$_1$ □ □ , then by Lemma3.1, T P: $_□$□$P_□$ is completely continuous.

We can obtain □$_t$ □ 0, t□[0,1] from □□ □ □( ) □ 0, □ □[,0] . Due to f^0 □ N_1, then there exists a constant r_1 □ 0 such that $f\,t(\,,$□$,\,)v$ □ N_1($||$□$||_{C1}$ □v) for □□C_1□, v□$\mathbb{R}$□ and $||$□$||_{C}$1 □ □v [0,r_1]. In the same way, because of $h\,N^0$ □ $_2$, then there exists a constant r_2 □ 0 such that $h\,t(\,,$□$)$ □ N_2 $||$□$||_{C}$1 for t□[0,1] , □□C_1□ and $||$□$||_{C}$1□[0,r_2] . Because of k^0 □ N_3, then there exists a constant r_3 □ 0such that $k\,t\,v(\,,\,)$ □ $N\,v_3$ for t□[0,1] and v□[0,r_3] . Due to g^0 □ N_4, then there exists a constant r_4 □ 0 such that $g\,v(\,)$ □ $N\,v_4$ for v□[0,r_4]. For convenience, we denote

□1 □ N1(1□q0) □ N2 □ □N3 N4.
□ □ □(□ □ 1) □ □(□ 1)

Let

□ r^1,r r r_2, $_3$, $_4$ □□, b□ □(1 □$_1$) .r_5
r_5 □ min□
□1□q_0 □

According to Lemma2.2, when0 □ $a\,h^{□\,(0,□)}$ □b , for any
□ □(□ 1)
□ □ □$_{r5}$ □z $P_□$:$||\,z\,||$□r_5□, z□□□$_{r5}$,
then $||\,z\,||$□r_5, we can get that
$||$□s □ zs $C||$ 1 □$||$ zs $C||$ 1□$||$ z $||$ □ r5,
s s

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$Qz\, s(\,)$ □□0 $q\, s(\,,\,)\,(\,)$d□ □ □z□ r_5 □0 $q\, s(\,,\,)$d□ □□ $q\, r_{0\,5}$

for s□[0,1], so □$_s$ □ □$z_s Qz\, s(\,)$ □ □$r_5 q\, r_{0\,5}$ □ r_1. Then, we have

1 1

$||Tz||$□ sup(□□0 $G\, t\, s\, f\, s(\,,\,)$ (,□$_s$ □ $z\, Qz\, s_s$, ( ))ds□□0 $H\, t\, s\, h\, s(\,,\,)$ (,□$_s$ □ z_s)ds

t [0,1]

□t□01$k\, s\, z\, s(\,,(\,))$ds□______□ □t(□□t□1) (a□h(0,□)) □ □(t t^2) _^{1}2 $g\, z((\,)))$□

1

□ sup(□□0 $G\, t\, s\, N(\,,\,)_1(||$□$_s$ □ $z_s||_{C1}$ □$q_0\, ||z||$)ds□ $N_3\, ||z||$ □$N_4\, ||z||$

t [0,1]

1

□□0 $H\, t\, s\, N(\,,\,)_2\, ||$□$_s$ □ $z_s||_{C1}$ ds

1 1

□ $N_1(1$□$q_0)\, ||z||$ □0 $p\, s_1(\,)$ds□ $N_2\, ||z||$ □0 $p\, s_2(\,)$ds□ $N_3\, ||z||$

□ $N_4\, ||z||$ □ ______a□h(0,□)

□ □(□ 1)

________ $N_1(1$□$q_0)||$ z $||$ □ N_2 ________ $||z||$ □$N_3\, ||z||$ □$N_4\, ||z||$ □ a□h(0,□)

□

□ □ □(□ □ 1) □ □(□ 1) □ □(□ 1)

$N_1(1$□$q_0)$ N_2 $N_3 N_4)$ $||z$ ________ $||$ □ a□h(0,□)

□ (□ □ □

□ □ □(□ □ 1) □ □(□ 1) □ □(□ 1)

□□$_{1\,5}r$ □ □b r_5,

So, for any z□P□□□$_r$5 , we get $||Tz||\;||$□ z ||.

Because of k□ □ N_5 , then there exists a constant r_6 □ r_5 such that $k\, t\, v(\,,\,)$ □ $N\, v_5$ for t□[0,1] and v□[0,(□□□ □□1)r_6]. For any □ □ □$_{r6}$ □$z\, P$□ :$||z||$□ r_6□, z□□□$_r$6 , we have $||z||$□r_6 . Then

1 1

$||Tz||$□ sup(□□0 $G\, t\, s\, f\, s(\,,\,)$ (,□$_s$ □ $z\, Qz\, s_s$, ( ))ds□□0 $H\, t\, s\, h\, s(\,,\,)$ (,□$_s$ □ z_s)ds

t [0,1]

□t□01$k\, s\, z\, s(\,,(\,))$ds□______□ □t(□□t□1) (a□h(0,□)) □ □(t t2) _12 $g\, z((\,)))$□

1

□□0 $k\, s\, z\, s(\,,(\,))$ds □ $N_5\, ||z||$ □ r_6,

hence, for any z□P□ □□$_r$6 , we get $||Tz||\;||$□ z ||.

In summary, from Lemma3.2, T has at least one fixed point z in P□(□ □$_{r6}$ \ $_r$5) , and0 □ □ □$r_5\, ||z||$ r_6 , so the boundary value problem (1.1) has at least one positive solution.

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